Light angular momentum flux and forces in birefringent inhomogeneous media

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The angular momentum carried by a monochromatic optical field is separated into an orbital and a spin part beyond the paraxial approximation. These quantities have been distinguished on the grounds of the different mechanical effects they produce in transparent and birefringent media endowed with internal degrees of freedom. The orbital and the spin angular momentum flux densities exhibited are shown to be divergence free in homogeneous and isotropic media and to give back the correct expressions in the paraxial limit.

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I. INTRODUCTION

The problem of the separation of the angular momentum of light in its intrinsic (or spin) and orbital parts has drawn the attention of many scientists in the past [1] and it has recently received a great deal of interest after the seminal paper by Allen *et al.* [2], but some confusion still remains especially beyond the paraxial optics approximation [3,4]. For further references on this subject, we address the reader to a recent review [5] and to the clear account by Crichton and Martson [6] on density of the radiative field. In the paraxial optics approximation, it is well known that the total angular momentum of monochromatic light can be decomposed uniquely into an orbital and a spin part and that Laguerre-Gauss modes carry a well defined orbital angular momentum [2,3]. In particular, the orbital angular momentum carried by a Laguerre-Gaussian beam is connected to its azimuthal angular dependence $\exp(il\phi)$, and the spin is connected to its polarization ellipticity σ_z. This was first issued from the formal analogy between the z components of the quantum-mechanical operators for the orbital and spin angular momenta and the two terms that, added together in the paraxial optics approximation, yield the total angular momentum per unit energy flux of a monochromatic wave [3]. Beyond the paraxial optics approximation, the total angular momentum density of the radiative field may be still calculated and split in its orbital and intrinsic parts [1,3], but the connection of l and σ_z with the orbital angular momentum and the polarization of the field is lost, in general. The analogy with quantum mechanics turns out to be useless in this case, since the quantum operators $\mathcal{L}_z$ and $\mathcal{S}_z$, acting on nonparaxial (near) cylindrically symmetric beams, mix terms depending on l and terms depending on σ_z, and the total angular momentum is no longer the sum of such terms only [7]. Very recently, it has been demonstrated that, without resorting to the paraxial optics approximation, the flux of the angular momentum carried by monochromatic beams can be separated into an orbital and a spin part, which are considered physically separated on the grounds of the different behavior they exhibit in propagating through an element that imparts on the beam an azimuthal dependence or through a birefringent plate [8]. Nevertheless, the orbital and spin angular momentum introduced in Ref. [8] come from two densities not obeying separate conservation laws and therefore are not separately divergence free in a homogeneous, nonabsorbing, and isotropic medium as vacuum. Consequently, the radiative angular momentum fluxes passing through an arbitrary closed surface would depend, in general, on the surface, which is impossible. Separately divergence free expressions for the orbital and spin angular momentum fluxes are desirable to investigate the separation of the total angular momentum flux not only for cylindrically symmetric optical fields but also for other radiation fields, such as, for example, the dipole field.

In principle, to separate the total angular momentum of light into meaningful orbital and spin parts, one should exhibit two quantities that, when added, yield the total angular momentum and that, in isotropic, nonabsorbing, and homogeneous media, are independently conserved.

To our knowledge all the works that appeared on this matter dealt with electromagnetic fields in transparent isotropic homogeneous media and the two parts of the total radiative angular momentum have been usually singled out on purely formal grounds. In the present work, we try to separate the orbital and the spin part of the total angular momentum carried by a monochromatic optical beam following a more physical approach, involving the interaction of the beam with anisotropic and inhomogeneous transparent media. In fact, the separation into an intrinsic and an orbital part of the angular momentum of such materials can be unambiguously performed. Liquid crystals, for example, are fluids made up of elongated molecules, whose centers of mass are randomly disposed. In appropriate temperature ranges, however, the molecules in each volume element dV located at a position $\boldsymbol{r}$ at a time t may assume a common average direction $\boldsymbol{n}(\boldsymbol{r},t)$, called the molecular director. Such materials have the merit, on the one hand, of being very sensitive to external optical as well as static magnetic and electrical fields [9–11] and, on the other hand, of having clearly distinguishable orbital and intrinsic degrees of freedom. A rotation of the director $\boldsymbol{n}(\boldsymbol{r},t)$ in the fixed volume element dV is associated with the intrinsic (spin) part of the angular momentum of the material, while a rotational motion of the center of mass $\boldsymbol{r}$ of dV is associated with the orbital part. It

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seems therefore natural to call the orbital and the spin angular momentum of the optical field the parts of the field producing a torque density acting on the fluid element dV or on the average orientation $\boldsymbol{n}(\boldsymbol{r},t)$ of the molecules contained in dV, respectively. Liquid crystals therefore turn out to be a very good arena to unambiguously define the $\hat{L}$ and $\hat{S}$ fluxes.

As a matter of fact, for a long time it has been shown that the spin angular momentum carried by a circularly polarized light beam can be transferred to a liquid crystal, putting its molecules into rotation around the propagation direction [12–14], as in the celebrated experiment made by Beth in 1936 using a quartz plate [15]. Very recent experiments, besides, proved the existence of a new source of torque acting on $\boldsymbol{n}(\boldsymbol{r},t)$ along the beam propagation direction [16–19]. This new torque cannot be related to the photon spin, since it was found to be still present when unpolarized light was used [16,18,20]. Rather, the new torque was found to be strongly dependent on the shape of the transverse profile of the incident light beam [16,18]. It was suggested that the origin of this torque could be retraced back to the orbital rather than the spin part of the photon angular momentum [18,19]. This guess was supported by approximate calculations made either assimilating the liquid crystal sample to a birefringent plate with a Gaussian shaped retardation profile in the transverse plane [18] or by exploiting some integral relationships coming from Noether's theorems applied to the liquid crystal free energy functional [19]. Although very approximate, both approaches lead to the idea that the spin part of the light angular momentum couples with $\boldsymbol{n}(\boldsymbol{r},t)$ directly, while the orbital part couples with the gradients of $\boldsymbol{n}(\boldsymbol{r},t)$. The combined effects due to the spin and the orbital parts of the light angular momentum are very impressive in liquid crystals, leading to complex dynamics of $\boldsymbol{n}$ [18,19] as well as to deterministic chaos and on-off rotational intermittence [20].

In the present work, the sources of the optical torques acting in the liquid crystalline material are retraced back in a rigorous way to the angular momentum fluxes carried by a monochromatic radiation in the surrounding medium, that is supposed to be homogenous and isotropic. The separation of the rotational motions inside the material and of the torques they arise from has repercussions on the corresponding fluxes in the radiation incident from the outside. As we shall see, although the torque densities are unambiguously determined, the corresponding flux densities still exhibit some ambiguity, since they are defined within a "gauge" transformation [see Eq. (9) below]. It is a remarkable result, however, that a gauge can be found where the flux densities of the orbital and spin parts of the optical angular momentum become both divergence free when a homogeneous and isotropic medium as vacuum is envisaged (the total angular momentum flux is divergence free in any gauge and in any medium). As a consequence, when this particular gauge is chosen, the cycle-averaged orbital and spin angular momenta of light in vacuum reduce to two independently conserved quantities. A sideways result of this approach is that the coupling of the orbital part of the light angular momentum with the gradients of $\boldsymbol{n}(\boldsymbol{r},t)$ is not direct, as previously supposed [16,18–20], but mediated by the rotational motion of the fluid. In the next section the equations of motion of matter under the action of forces and momenta generated by electromagnetic fields are derived in the continuum theory framework. Internal rotational degrees of freedom of the medium are accounted for. In Sec. III, the flux densities of the electromagnetic angular momenta are derived for monochromatic fields and a separation between the orbital and spin angular momentum fluxes is proposed. In Sec. IV our definitions of the orbital and the spin angular momentum fluxes are applied to the problem of dipole radiation. Finally, in Sec. V our conclusions are drawn.

II. DYNAMICAL EQUATIONS

We start from the usual equations of motion for the liquid crystalline fluid and for the director $\boldsymbol{n}$:

$$\rho\dot{\mathbf{v}} = \boldsymbol{f} = \mathrm{div}\hat{\sigma} - \mathrm{grad}\, p, \tag{1a}$$

$$\rho\boldsymbol{r}\times\dot{\mathbf{v}} = \boldsymbol{r}\times\boldsymbol{f} = \mathrm{div}\hat{L} - \boldsymbol{w}, \tag{1b}$$

$$I\boldsymbol{n}\times\ddot{\boldsymbol{n}} = \boldsymbol{\tau} = \mathrm{div}\hat{S} + \boldsymbol{w}, \tag{1c}$$

where ρ is the fluid density (assumed to be constant), $\mathbf{v}$ is the velocity of the fluid (flow of the centers of mass of dV), $\boldsymbol{f}$ is the force per unit volume acting on the fluid, p is the hydrostatic pressure, I is the momentum of inertia per unit volume associated to the rotation of $\boldsymbol{n}$, $\boldsymbol{\tau}$ is the torque density acting on $\boldsymbol{n}$, and, finally $\hat{\sigma}$ is the stress tensor and $\boldsymbol{w}$ is the vector dual to its antisymmetric part, i.e., $w_\alpha = \epsilon_{\alpha\beta\gamma}\sigma_{\beta\gamma}$. Elastic (e), electromagnetic (em), and viscous (v) forces contribute, in general, to the torque density $\boldsymbol{\tau} = \boldsymbol{\tau}^e + \boldsymbol{\tau}^{em} + \boldsymbol{\tau}^v$ and to the stress tensor $\hat{\sigma} = \hat{\sigma}^e + \hat{\sigma}^{em} + \hat{\sigma}^v$. The tensor $\hat{L}$ is defined as $L_{\rho\alpha} = \epsilon_{\alpha\beta\gamma}x_\beta\sigma_{\rho\gamma}$ so that Eq. (1b) follows from Eq. (1a). The tensor $\hat{S}$, on its hand, comes from a variational principle based on a suitable free energy functional from which the torque density $\boldsymbol{\tau}$ and the stress tensor $\hat{\sigma}$ can be also deduced. The divergence of a tensor $\hat{T}$ here is defined as $(\mathrm{div}\hat{T})_\alpha = \partial_\rho T_{\rho\alpha}$. The terms on the left in Eqs. (1) may be unambiguously interpreted as the densities per unit time of linear momentum, orbital angular momentum, and intrinsic angular momentum of matter, respectively. The terms on the right of Eqs. (1) can be consistently interpreted as the densities of force, orbital torque, and intrinsic torque acting in the bulk of the medium. Such densities are represented by expressions that contain the divergences of tensors involving the external fields and then make evident their relationship with the fluxes of force and momentum from the outside. It is therefore quite natural referring to the tensors $\hat{L}$ and $\hat{S}$ in Eqs. (1b) and (1c) as to the orbital and the intrinsic (spin) angular momentum flux densities, respectively. On this definition of fluxes we will return later. In the meantime, let us calculate the stress tensor $\hat{\sigma}$, the torque density $\boldsymbol{\tau}$, and the intrinsic angular momentum flux density $\hat{S}$. The elastic and electromagnetic contributions can be deduced applying variational calculus to the free energy functional $\mathcal{F} = \int_V F\, dV = \int_V (F^e + F^{em})dV$, where as densities of the elastic and electromagnetic free energy we may take, respectively,

$$F^e = \frac{1}{2}[k_1(\mathrm{div}\ \boldsymbol{n})^2 + k_2(\boldsymbol{n}\cdot \mathrm{rot}\ \boldsymbol{n})^2 + k_3(n\times \mathrm{rot}\ \boldsymbol{n})^2], \quad (2)$$

where k_i $(i=1,2,3)$ are the elastic constants for splay, twist, and bend deformations, and

$$F^{em} = \frac{1}{16\pi}(\boldsymbol{B}^*\cdot\boldsymbol{H} - \boldsymbol{D}^*\cdot\boldsymbol{E}), \quad (3)$$

where monochromatic optical fields are assumed and the magnetic and electric inductions $\boldsymbol{B}$ and $\boldsymbol{D}$ are related to the corresponding fields by $\boldsymbol{B}=\hat{\mu}\boldsymbol{H}$, $\boldsymbol{D}=\hat{\epsilon}\boldsymbol{E}$, with magnetic and dielectric tensors related to $\boldsymbol{n}$ by the uniaxial form: $\hat{\mu}=\mu_0 + \mu_a\boldsymbol{nn}$, $\hat{\epsilon}=\epsilon_0+\epsilon_a\boldsymbol{nn}$. The constants μ_0, ϵ_0, μ_a, ϵ_a characterize the magnetic and electric response of the material. In particular, μ_a and ϵ_a characterize the material anisotropy and they vanish in isotropic media. For monochromatic optical fields, $\boldsymbol{B}$ and $\boldsymbol{H}$ can be related to the spatial derivatives of the electric field $\boldsymbol{E}$, using Maxwell's equation and constitutive relation

$$\boldsymbol{B} = -(i/k_0)\mathrm{rot}\boldsymbol{E}, \quad \boldsymbol{H} = \hat{\eta}\boldsymbol{B}, \quad (4)$$

with $k_0=\omega/c$, c being the speed of light in vacuum and ω the optical frequency (cgs units are used), and $\hat{\eta}=\hat{\mu}^{-1}=\eta_0 + \eta_a\boldsymbol{nn}$. Inserting Eqs. (4) into Eq. (3), the total free energy density $F=F^e+F^{em}$ reduces to a function of the fields $\boldsymbol{n}(\boldsymbol{r})$, $\boldsymbol{E}(\boldsymbol{r})$, $\boldsymbol{E}^*(\boldsymbol{r})$, and of their spatial derivatives. The field equations associated to the total free energy $\mathcal{F}$ are

$$\boldsymbol{h} = \mathrm{div}\hat{\pi} - \partial F/\partial\boldsymbol{n} = \lambda(\boldsymbol{r})\boldsymbol{n}, \quad (5a)$$

$$\boldsymbol{\Lambda} = \mathrm{div}\hat{p} - \partial F/\partial\boldsymbol{E}^* = \boldsymbol{0}, \quad (5b)$$

where $\pi_{\rho\gamma}=\partial F/\partial(\partial_\rho n_\gamma)$ and $p_{\rho\gamma}=\partial F/\partial(\partial_\rho E^*_\gamma)$ are the tensors of the generalized momenta associated to the fields $\boldsymbol{n}$ and $\boldsymbol{E}^*$, respectively, and $\lambda(\boldsymbol{r})$ is a Lagrange multiplier accounting for the constraint $\boldsymbol{n}^2=1$. At steady state ($\boldsymbol{v}=\boldsymbol{0}$, $\dot{\boldsymbol{n}}=\boldsymbol{0}$), Eq. (5a) is equivalent to Eq. (1c), the sum of the elastic and of the electromagnetic torque densities being given by $\boldsymbol{\tau}^e+\boldsymbol{\tau}^{em}=\boldsymbol{n}\times\boldsymbol{h}$. Equation (5b), on the other hand, is equivalent to Maxwell's equation $\mathrm{rot}(\hat{\eta}\mathrm{rot}\ \boldsymbol{E})=k_0^2\boldsymbol{D}$. The elastic and electromagnetic contributions $\hat{\sigma}^e$ and $\hat{\sigma}^{em}$ to the stress tensors $\hat{\sigma}$ in Eq. (1a) are given by the opposite of the energy-momentum tensors associated to F^e and F^{em}, respectively, i.e.,

$$\hat{\sigma}^e_{\beta\alpha} = -\pi_{\beta\gamma}\partial_\alpha n_\gamma + \delta_{\beta\alpha}F^e, \quad (6a)$$

$$\hat{\sigma}^{em}_{\beta\alpha} = -p_{\beta\gamma}\partial_\alpha E^*_\gamma + \delta_{\beta\alpha}F^{em}. \quad (6b)$$

It can be easily proved that the electromagnetic force $\boldsymbol{f}^{em}=\mathrm{div}\ \hat{\sigma}^{em}$ acting on the unit volume has the right form [21] $f^{em}_\alpha=-(1/16\pi)(E^*_\beta E_\gamma\partial_\alpha\epsilon_{\beta\gamma}+H^*_\beta H_\gamma\partial_\alpha\mu_{\beta\gamma})$. In the liquid crystal community $\boldsymbol{h}$ is known as the molecular field. The explicit expression of the elastic contributions to $\boldsymbol{h}$ can be found in standard textbooks on the physics of liquid crystals [see, for example, Ref. [9], Eq. (3.22)]. The electromagnetic contribution $\boldsymbol{\tau}^{em}$ to the torque density $\boldsymbol{\tau}$ results in the sum of the optical torque $\boldsymbol{\tau}^o=1/(8\pi)\mathrm{Re}(\boldsymbol{D}^*\times\boldsymbol{E})$ and of the magnetic torque $\boldsymbol{\tau}^{mag}=1/(8\pi)\mathrm{Re}(\boldsymbol{B}^*\times\boldsymbol{H})$. At optical frequencies, liquid crystals are nonmagnetic, $\boldsymbol{B}$ and $\boldsymbol{H}$ are parallel, and $\boldsymbol{\tau}^{mag}$ vanishes, leaving only the optical torque $\boldsymbol{\tau}^o$. In the presence of radiative electromagnetic fields, where the optical cycle-averaged magnetic and electric energy densities are equal, the total free energy $\mathcal{F}$ takes the same value as its elastic part only, thus reaching a true minimum at equilibrium. The invariance of $\mathcal{F}$ with respect to a rotation of the coordinate frame may be exploited to split both the elastic and the electromagnetic torque density into the sum of the divergence of a tensor and the antisymmetric part of the corresponding stress tensor,

$$\boldsymbol{\tau}^h = \mathrm{div}\ \hat{S}^h + \boldsymbol{w}^h \quad (h=e,em). \quad (7)$$

The identity in Eq. (7) holds true for an arbitrary field $\boldsymbol{n}$ and a field $\boldsymbol{E}$ obeying Maxwell's equations (5b). The tensors $\hat{S}^e$ and $\hat{S}^{em}$ can be regarded as the elastic and the electromagnetic "spin flux densities," respectively.

The contribution σ^v of viscous forces to the overall stress tensor can be deduced on the grounds of phenomenological considerations and can be found in textbooks on the physics of liquid crystals [see, for example, Ref. [9], Eqs. (5.27) and (5.28)]. Adding the electromagnetic field does not change $\hat{\sigma}^v$, since no entropy source is associated to the optical field, when light absorption is neglected. In particular, we still have the useful relationship $\boldsymbol{\tau}^v=\boldsymbol{w}^v$ between the viscous torque density and the antisymmetric part of $\hat{\sigma}^v$. Comparing this relationship with Eq. (7), we conclude that no "spin flux density" is associated to viscous torques. From Eq. (7) and from the relation $\boldsymbol{\tau}^v=\boldsymbol{w}^v$, we see that the last equality on the right of Eq. (1c) is a consequence of the rotational invariance of the total free energy of the system.

III. ANGULAR MOMENTUM FLUXES

Adding Eqs. (1b) and (1c) together yields

$$\rho\boldsymbol{r}\times\dot{\boldsymbol{v}} + I\boldsymbol{n}\times\ddot{\boldsymbol{n}} = \mathrm{div}(\hat{L}+\hat{S}) = \mathrm{div}\hat{J}, \quad (8)$$

stating the conservation of the total (orbital + intrinsic) angular momentum of the system. The fluxes $\hat{L}$ and $\hat{S}$ do not conserve separately, however, because of the presence of the vector $\boldsymbol{w}$ in Eqs. (1b) and (1c). The vector $\boldsymbol{w}$ is to be interpreted as an internal torque in the volume element dV which couples the L- and S-flux densities. If the total stress tensor was symmetric, then $\boldsymbol{w}$ would vanish and $\hat{L}$ and $\hat{S}$ would exhibit separate conservation laws. The stress tensor $\sigma_{\rho\alpha}$ and the related L- and S-flux densities, however, are determined up to the following gauge transformations:

$$\sigma_{\rho\alpha}\rightarrow\sigma'_{\rho\alpha}=\sigma_{\rho\alpha}+\partial_\gamma f_{\gamma\rho\alpha},$$

$$L_{\rho\alpha}\rightarrow L'_{\beta\alpha}=\epsilon_{\alpha\beta\gamma}\chi_\beta\sigma'_{\rho\gamma},$$

$$S_{\rho\alpha}\rightarrow S'_{\rho\alpha}=S_{\rho\alpha}+\epsilon_{\alpha\beta\gamma}f_{\rho\beta\gamma}, \quad (9)$$

where $f_{\gamma\rho\alpha}=-f_{\rho\gamma\alpha}$. Equations (1) are invariant under the transformations (9). The gauge function $f_{\gamma\rho\alpha}$ may be

TABLE I. Stress tensor and intrinsic angular momentum flux in different gauges. The flux of the orbital angular momentum is given by $L_{\alpha\beta}=\epsilon_{\beta\mu\nu}x_\mu\sigma_{\alpha\nu}$. The several contributions, for each block, are labeled as S for splay, T for twist, and B for bend, corresponding to the fundamental elastic distortion in nematic liquid crystals. The label E represents the elastic contribution as a whole and em is for the electromagnetic contribution. Finally, we posed $A=\boldsymbol{n}\cdot\mathrm{rot}\,\boldsymbol{n}$; $\boldsymbol{B}=\boldsymbol{n}\times\mathrm{rot}\,\boldsymbol{n}$.

		$\sigma_{\alpha\beta}$	$\boldsymbol{w}$	$S_{\alpha\beta}$
I	S	$n_\alpha h^S_\beta-\delta_{\alpha\beta}(F^S+\boldsymbol{n}\cdot\boldsymbol{h}^S)$	$\boldsymbol{n}\times\boldsymbol{h}^S$	0
	T	$-n_\beta h^T_\alpha-\delta_{\alpha\beta}F^T$	$\boldsymbol{n}\times\boldsymbol{h}^T$	0
	B	$-n_\beta h^B_\alpha-k_3B_\alpha B_\beta$	$\boldsymbol{n}\times\boldsymbol{h}^B$	0
		$-\left(n_\alpha\partial F^B/\partial n_\beta+n_\beta\partial F^B/\partial n_\alpha\right)+\delta_{\alpha\beta}F^B$		
	em	$1/16\pi[(D^*_\alpha E_\beta+B^*_\alpha H_\beta+\mathrm{c.c.})$	$1/16\pi(\boldsymbol{D}^*\times\boldsymbol{E}+\boldsymbol{B}^*\times\boldsymbol{H}+\mathrm{c.c.})$	0
		$-\delta_{\alpha\beta}(\boldsymbol{D}^*\cdot\boldsymbol{E}+\boldsymbol{B}^*\cdot\boldsymbol{H})]$		
II	S	$-k_1\partial_\rho n_\rho\partial_\beta n_\alpha+\delta_{\alpha\beta}F^S$	$k_1\mathrm{rot}\,\boldsymbol{n}\,\mathrm{div}\,\boldsymbol{n}$	$k_1\partial_\rho n_\rho\epsilon_{\alpha\beta\gamma}n_\gamma$
	T	$-k_2A\epsilon_{\alpha\gamma\rho}n_\rho\partial_\beta n_\gamma+\delta_{\alpha\beta}F^T$	$-k_2A(\boldsymbol{B}+\boldsymbol{n}\,\mathrm{div}\,\boldsymbol{n})$	$k_2A(n_\alpha n_\beta-\delta_{\alpha\beta})$
	B	$-k_3(n_\gamma B_\alpha-n_\alpha B_\gamma)\partial_\beta n_\gamma+\delta_{\alpha\beta}F^B$	$-k_3\{[(\boldsymbol{n}\times\boldsymbol{B})\cdot\nabla]\boldsymbol{n}-(\boldsymbol{n}\times\boldsymbol{B})\mathrm{div}\,\boldsymbol{n}\}\}$	$k_3n_\alpha\epsilon_{\beta\gamma\rho}B_\gamma n_\rho$
	em	$i/16\pi k_0\,\epsilon_{\alpha\gamma\rho}(H^*_\rho\partial_\beta E_\gamma-\mathrm{c.c.})+\delta_{\alpha\beta}F^{em}$	$i/16\pi k_0[\boldsymbol{H}^*\mathrm{div}\,\boldsymbol{E}-(\boldsymbol{H}^*\cdot\nabla)\boldsymbol{E}-\mathrm{c.c.}]]$	$i/16\pi k_0(-H^*_\beta E_\alpha+\delta_{\alpha\beta}\boldsymbol{H}^*\cdot\boldsymbol{E}-\mathrm{c.c.})$
III	E	$-K(\partial_\alpha n_\gamma\partial_\beta n_\gamma-\frac{1}{2}\delta_{\alpha\beta}\partial_\gamma n_\rho\partial_\gamma n_\rho)$	$\delta k_1\boldsymbol{\omega}^S+\delta k_2\boldsymbol{\omega}^T+\delta k_3\boldsymbol{\omega}^B$	$K\epsilon_{\beta\gamma\rho}n_\gamma\partial_\alpha n_\rho$
		$+\delta k_1\sigma^S_{\alpha\beta}+\delta k_2\sigma^T_{\alpha\beta}+\delta k_3\sigma^B_{\alpha\beta}$		$+\delta k_1S^S_{\alpha\beta}+\delta k_2S^T_{\alpha\beta}+\delta k_3S^B_{\alpha\beta}$
	em	$-1/16\pi k_0^2\{\eta_o(\partial_\alpha E_\gamma\partial_\beta E^*_\gamma-\partial_\rho E_\rho\partial_\beta E^*_\alpha+\mathrm{c.c.})$	$-\eta_o/16\pi k_0^2\mathrm{rot}\,\boldsymbol{E}^*\mathrm{div}\,\boldsymbol{E}+\mathrm{c.c.}$	$\eta_o/16\pi k_0^2(\epsilon_{\beta\gamma\rho}E^*_\gamma\partial_\alpha E_\rho$
		$-\delta_{\alpha\beta}[\eta_o(\partial_\gamma E_\rho\partial_\gamma E^*_\rho-\partial_\rho E_\rho\partial_\gamma E^*_\gamma)-k_0^2\boldsymbol{D}^*\cdot\boldsymbol{E}]\}$		$-\epsilon_{\alpha\beta\gamma}E^*_\gamma\partial_\rho E_\rho)+\mathrm{c.c.}$

uniquely chosen so to have the components of the spin flux tensor $S'_{\rho\alpha}$ arbitrarily fixed. In particular, the gauge may be fixed so that $S'_{\rho\alpha}=0$. In this gauge we have $\boldsymbol{\tau}=\boldsymbol{n}\times\boldsymbol{h}=\boldsymbol{w}$, which means that the torque acting on $\boldsymbol{n}$ is fully determined by the antisymmetric part of the stress tensor. Moreover, in the spinless gauge the orbital and the total angular momentum flux densities are the same, i.e., $\hat{L}=\hat{J}$. This spinless gauge is commonly exploited in the physics of fluids to symmetrize the stress tensor: assuming, in fact, the intrinsic angular momentum to be locally balanced, i.e., $\boldsymbol{\tau}=\boldsymbol{0}$, then, the antisymmetric part of the stress tensor $\boldsymbol{w}$ turns to be zero, yielding to a totally symmetric stress tensor $\hat{\sigma}$. The condition of balance of the torques acting on $\boldsymbol{n}$ entails that the inertial term on the left of Eq. (1c) is zero or negligible, as usually assumed in liquid crystals. The last peculiarity was exploited by the Harvard group long ago to describe the hydrodynamics of liquid crystals through a symmetric stress tensor in the small elastic distortion approximation [22]. In block I of Table I, we have reported the stress tensor, its antisymmetric part, and the spin flux density tensor in the spinless gauge. In this gauge $\hat{S}=0$, by definition, and $\boldsymbol{\tau}=\boldsymbol{w}$. If we further assume $\boldsymbol{\tau}=\boldsymbol{0}$, we may retain, in calculating the force density $\boldsymbol{f}$, only the symmetric part of the total stress tensor $\hat{\sigma}$ reported in the block I of Table I. In particular, in the spinless gauge, the electromagnetic part of the force density $\boldsymbol{f}$ reduces to the divergence of the symmetric part of Maxwell's stress tensor $\hat{\sigma}^M$ as it holds true in ordinary crystals [23]. Though useful to simplify some calculations on slightly distorted liquid crystals, the spinless gauge presents some drawbacks: the dynamical constraint $\boldsymbol{\tau}=\boldsymbol{0}$ (the local balance of the angular momentum), in fact, is not generally satisfied and, what is worse, the intrinsic and the orbital parts of the angular momentum flux in the material mix so as to become unrecognizable. On the contrary, Ericksen's traditional approach, which is based on the free energy densities in Eqs. (2) and (3), keeps the orbital and spin angular momenta separated, and seems therefore physically more appropriate, though leading to a nonsymmetric stress tensor [24]. Ericksen's stress tensor, its antisymmetric part, and Ericksen's spin tensor are reported in block II of Table I. It is worth noting that the definitions of the orbital and spin angular momentum flux along the z axis for a monochromatic field in vacuum proposed in Ref. [8] can be brought back just to the L_{33} and S_{33} elements reported in the block II of Table I. Here we derived the same flux densities from a more general Lagrangian approach, exploiting the rotational symmetry of the system. The main drawback of the electromagnetic flux densities $\hat{L}^{em}$ and $\hat{S}^{em}$ derived from the Lagrangian in Eq. (3) is that they are not divergence free even in vacuum (only $\hat{J}^{em}=\hat{L}^{em}+\hat{S}^{em}$ is divergence free in vacuum). Having divergenceless fluxes $\hat{L}^{em}$ and $\hat{S}^{em}$ is desirable to have separate conservation laws for the orbital and spin angular momenta.

In birefringent media such as liquid crystals, the stress tensor $\hat{\sigma}$ is not symmetric, in general, even in the spinless gauge and the internal torque $\boldsymbol{w}$ in Eqs. (1b) and (1c) is present also in that case. The presence of the internal torque $\boldsymbol{w}$ is due to the lack of invariance of the total free energy $\mathcal{F}$ of the system under separate rotation of the center of mass $\boldsymbol{r}$ and of the components of the fields $\boldsymbol{n}$ and $\boldsymbol{E}$. However, the elastic free energy becomes rotationally invariant when all elastic constants k_i $(i=1,2,3)$ become equal, so we may expect that the stress tensor will be symmetric in this limit. Setting $k_i=K$ in Eq. (2), F_e reduces to

$$F^0=\frac{K}{2}[(\mathrm{div}\,\boldsymbol{n})^2+(\mathrm{rot}\,\boldsymbol{n})^2]. \tag{10}$$

The stress tensor $\hat{\sigma}^0$ derived from F^0 is still nonsymmetric. However, F^0 differs from the free energy density F^1

$=(K/2)\partial_\alpha n_\beta \partial_\alpha n_\beta$ by divergence terms only, so that F^0 and F^1 are equivalent in the bulk [see Ref. [9], Eq. (3.17)], but the stress tensor $\hat{\sigma}^1$ derived from F^1 is now symmetric. We may write the original elastic free energy density F^e as $F^e = F^0 + \tilde{F}^e$, where $\tilde{F}^e$ is obtained from F^e through the formal substitution $k_i \rightarrow (k_i - K)/k_i$ $(i=1,2,3)$. By this choice, when all elastic constants tend to the common value K (this may be the case in liquid crystals near the nematic to isotropic transition), $F^e \rightarrow F^0$. Using this decomposition and exploiting the equivalence between F^0 and F^1 we may construct a new stress tensor that, though nonsymmetric in general, becomes symmetric in the one elastic constant approximation ($\delta k_i \rightarrow 0$). This elastic stress tensor and the corresponding spin flux are reported in block III of the Table I. A similar argument can be applied to write, within divergence terms, the electromagnetic free energy density F^{em} as $F^1_{em} + \tilde{F}_{em}$, where F^1_{em} generates a symmetric stress tensor in isotropic and homogenous media and $\tilde{F}_{em}$ represents the contribution from the optical anisotropy. Because the electromagnetic stress tensor associated to F^1_{em} is symmetric and $\tilde{F}_{em}$ vanishes in isotropic media, we obtain a stress tensor that reduces to a symmetric one in homogeneous isotropic media. This choice for F^{em} leads to the quantities listed in the last row of block III. We notice that, when this gauge is used, the antisymmetric part $\boldsymbol{w}$ of the stress tensor is proportional to div $\boldsymbol{E}$, which is zero in homogeneous and isotropic media. The symmetry of the stress tensor in such media entails that the corresponding flux densities $\hat{L}$ and $\hat{S}$ are both divergence free ($\boldsymbol{f}$ and $\boldsymbol{\tau}$ are also zero). It is remarkable that there is no gauge function like $f_{\gamma\rho\alpha}$ in Eqs. (9) settling the crossing between gauges I or II to the last one. This is not surprising, considering that the gauge transformations in Eq. (9) are not the most general: we may still add to $\sigma_{\rho\alpha}$ a divergence free *symmetric* tensor. Assuming now the liquid crystal sample to be immersed in a homogenous and isotropic medium, Eqs. (1b) and (1c), upon integration over a region V with its border ∂V completely immersed in the surrounding medium, assume the form

$$\int_V \rho(\boldsymbol{r}\times\dot{\boldsymbol{v}})\cdot\boldsymbol{u}\,d\boldsymbol{r} = \oint_{\partial V}\boldsymbol{u}\cdot\hat{L}^{em}\cdot\boldsymbol{u}\;ds - \int_V \boldsymbol{w}^{em}\cdot\boldsymbol{u}\;dr - \int_V \boldsymbol{w}^{e}\cdot\boldsymbol{u}\;dr + \int_V (\boldsymbol{r}\times\boldsymbol{f}^{v})\cdot\boldsymbol{u}\,d\boldsymbol{r}, \tag{11a}$$

$$\int_V I(\boldsymbol{n}\times\ddot{\boldsymbol{n}})\cdot\boldsymbol{u}\,d\boldsymbol{r} = \oint_{\partial V}\boldsymbol{u}\cdot\hat{S}^{em}\cdot\boldsymbol{u}\;ds + \int_V \boldsymbol{w}^{em}\cdot\boldsymbol{u}\;dr + \int_V \boldsymbol{w}^{e}\cdot\boldsymbol{u}\;dr + \int_V \boldsymbol{\tau}^{v}\cdot\boldsymbol{u}\;dr. \tag{11b}$$

When the quantities defined in block III are used, the internal torque $\boldsymbol{w}^e$ in Eqs. (11) vanishes in the one elastic constant approximation and the internal torque $\boldsymbol{w}^{em}$ vanishes in homogeneous and isotropic media. In deriving Eqs. (11) we assumed $\boldsymbol{v}=\boldsymbol{0}$ and appropriate anchoring conditions of $\boldsymbol{n}$ at the sample walls so to have no surface contribution from the elastic intrinsic angular momentum flux $\hat{S}^e$. An example of such anchoring conditions is a nematic film with homeotropic alignment at the walls as used in the experiments [18–20].

The flux densities $\hat{L}^{em}$ and $\hat{S}^{em}$ in Eqs. (11) are evaluated in the surrounding isotropic homogeneous medium, where they are both divergence free. The closed surface ∂V is therefore essentially arbitrary and the surface integrals in Eqs. (11) can be well identified with the fluxes of L and S coming from the external optical field. The two fluxes are physically discriminated in Eqs. (11) on the grounds of the different mechanical effects they produce in the medium, so that $\hat{L}^{em}$ and $\hat{S}^{em}$ can be identified as the flux densities of the orbital and intrinsic angular momentum carried by the optical field through the surface ∂V, respectively. We emphasize that the gauge leading to Eqs. (11) (block III of Table I) has been selected from the infinite possible ones because it is the only one leading to angular momentum flux densities $\hat{L}^{em}$ and $\hat{S}^{em}$ both conservative in isotropic and homogeneous media. In the light of such interpretation, Eqs. (11) show how the angular momentum is transferred from the external optical field to the liquid crystal. In the physics of liquid crystals, however, the inertial terms on the left of Eqs. (1) and (11) are usually neglected and the equations are solved with respect to the viscous torques and forces, that are proportional to $\dot{\boldsymbol{n}}$, to the gradients of $\dot{\boldsymbol{n}}$, and to the fluid velocity $\boldsymbol{v}$. In most cases the fluid motion can be also neglected. Then, setting $\boldsymbol{v}\approx 0$ in Eqs. (11) yields to two integral relationships involving only $\boldsymbol{n}$ and its time and space derivatives. A closer inspection shows that Eq. (11a) couples $\hat{L}^{em}$ to the space derivatives of $\boldsymbol{n}$, while Eq. (11b) couples $\hat{S}^{em}$ to $\boldsymbol{n}$ itself. All these features reproduce what was claimed in previous works where the plane wave approximation was adopted [25] or where approximate models were proposed to describe the effects of the orbital angular momentum of light in liquid crystals in Refs. [18–20].

IV. CASE OF DIPOLE RADIATION

The definitions of the optical fluxes $\hat{L}^{em}$ and $\hat{S}^{em}$ given in the last row of Table I apply beyond the paraxial optics approximation and, when such approximation is envisaged, they reduce, in homogenous and isotropic media, to the well-known expressions used to introduce the Laguerre-Gauss optical beams [3,8]. The relevance of handling divergence free quantities may be better understood on the grounds of an example: the dipole radiation. A straightforward calculation shows that the total angular momentum irradiated per unit time by a rotating dipole, evaluated with respect to its center of mass, is [1,8]

$$\boldsymbol{J} = \frac{ik_0^3}{3}(\boldsymbol{p}^*\times\boldsymbol{p}), \tag{12}$$

and that, according to the definitions in block III of Table I,

exactly half of it can be attributed to spin and half to the orbital angular momentum, i.e.,

$$\boldsymbol{S} = \boldsymbol{L} = \frac{ik_0^3}{6}(\boldsymbol{p}^* \times \boldsymbol{p}), \tag{13}$$

where $\boldsymbol{J}$, $\boldsymbol{L}$, and $\boldsymbol{S}$, respectively are the total, the orbital, and the spin angular momentum vectors irradiated by the dipole all over the solid angle. This result holds whatever the surface through which the flux is calculated and is in agreement with the suggestion made in Ref. [6]. If instead the flux densities reported in block II of Table I were used, $\boldsymbol{L}$ and $\boldsymbol{S}$ would depend on the closed surface chosen for the integration. For instance, the z components of the spin and orbital angular momentum irradiated per unit time through an ellipsoid with the dipole located at one of its focuses would be, in this case,

$$L_z = ik_0^3\left[\frac{1}{4e^2} - \frac{1}{8e^3}(1-e^2)\ln\frac{1+e}{1-e}\right](\boldsymbol{p}^* \times \boldsymbol{p})_z, \tag{14}$$

and

$$S_z = \frac{ik_0^3}{3}(\boldsymbol{p}^* \times \boldsymbol{p})_z - L_z, \tag{15}$$

which depend on the ellipticity e of the ellipsoid. This result rules out the possibility of using the fluxes $\hat{L}^{em}$ and $\hat{S}^{em}$ derived from the Lagrangian in Eq. (3) and proposed in previous works [8] as representative of the fluxes of angular momentum radiated by the dipole.

V. CONCLUSIONS

In conclusion, we studied the transfer of angular momentum from a monochromatic optical field to birefringent medium endowed with internal orientational degrees of freedom as, for example, liquid crystals. We were able to construct two electromagnetic fluxes $\hat{L}^{em}$ and $\hat{S}^{em}$ both conservative in vacuum (or in homogeneous isotropic media) which couples with the orbital and the intrinsic part of the angular momentum of matter, respectively (see block III of Table I). We could therefore identify $\hat{L}^{em}$ and $\hat{S}^{em}$ on physical rather than mathematical grounds as the orbital and intrinsic angular momentum fluxes carried by the optical field. The fluxes $\hat{L}^{em}$ and $\hat{S}^{em}$ reduce to well known expressions in the paraxial optics approximation. The present theory was carried out retaining full Maxwell's equations for the optical field in matter and inertial matter terms too. Moreover, no dynamical assumptions were made as, for example, the local torque balance as in Ref. [22]. When the inertial terms are neglected and appropriate approximations are made, our results confirm previous models proposed to explain the coupling of $\hat{L}^{em}$ to the gradients of $\boldsymbol{n}(\boldsymbol{r})$ [18–20]. We studied also the elastic problem in an analogous way and we were able to introduce a different elastic stress tensor (and associated angular momentum fluxes) which becomes symmetric when all elastic constants become equal, as it happens near the nematic clearing point. Finally, the drawbacks of other expressions reported sometimes in the literature for the stress tensor and angular momentum fluxes have been briefly discussed, exploiting the example of the dipole radiation. In this case we found that half of the angular momentum radiated by the dipole is orbital and half is intrinsic, as suggested in Refs. [1,6].

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